GENERAL DEMONSTRATIONS *of the* THEOREMS *for the* SINES *and* COSINES *of* MULTIPLE CIRCULAR ARCS, *and alſo of the* THEOREMS *for expreſſing the* POWERS *of* SINES *and* COSINES *by the* SINES *and* COSINES *of* MULTIPLE ARCS; *to which is added a* THEOREM *by help whereof the ſame* METHOD *may be applied to demonſtrate the* PROPERTIES *of* MULTIPLE HYPERBOLIC AREAS. *By the*

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THEOREMS by help of which the chords of multiple circular arcs may be found in terms of the chord of the ſimple arc were firſt given by Vieta, and afterwards in a different manner by Mr. Briggs, which are very fully explained in the Trigonometria Britannica, and their uſes in conſtructing trigonometrical tables ſhewn. From theſe may readily be deduced theorems for the coſines of multiple arcs in terms of the coſine of the ſimple arc, and for the ſines in terms of the ſine of the ſimple arc when the multiplier is an *odd* number, and conſequently the ſeries firſt given by Sir Iſaac Newton for the ſine of a multiple arc when the multiplier is an odd number, the only caſe in which that ſeries terminates—Afterwards ſimilar

theorems for the ſine and coſine of multiple arcs, when the multiplier is any whole poſitive number even or odd, were given by ſeveral authors—But all the writers on this ſubject that I have ſeen, except Dr. Waring, have deduced the law of the ſeries from obſervation in a few inſtances without a general demonſtration of its truth——Dr. Waring has (Curv. algebr. Propr Theor. 26 & Cor.) by help of his admirable theorem for finding the ſums of the powers of the roots of an equat. given a general demonſtration of the ſeries for finding the chord of the ſupplement of a multiple arc in terms of the chord of the ſupplement of the ſimple arc, and conſequently a general demonſtration of the theorem for the coſine of a multiple arc in terms of the coſine of the ſimple arc, and alſo of the ſine of a multiple arc when the multiplier is an odd number. But in the caſe where the multiplier is an even number no demonſtration, as far as I have ſeen, has ever been given by any author. Dr. Waring's method of demonſtration cannot be applied to this caſe—The following demonſtration extends to every multiplier whether even or odd. The demonſtrations for the ſine and coſine of the multiple arc in terms of the coſine of the ſimple arc, from whence the other theorems are immediately deducible, are of this kind—The probable law is deduced from obſervation in a few inſtances and then the general truth of that conjecture is proved. Dr. Waring's demonſtration, although by a very different proceſs, being founded upon the properties of algebraical equations, is alſo of this kind, as it depends

upon

upon his theorem for the ſums of the powers of the roots of an equation, of which he has given the ſame kind of demonſtration—Previous to the demonſtrations of theſe theorems I have given a demonſtration of the theorems for expreſſing the ſine and coſine of multiple arcs in terms compounded of the ſine and coſine—Theſe theorems alſo have been given by many authors, and the only general demonſtrations of them have been deduced from the hyperbola and the conſideration of impoſſible quantities—However uſeful impoſſible quantities may be in diſcovering mathematical truths they ought never to be uſed in ſtrict demonſtration, and it muſt ſeem a very circuitous mode to apply the properties of the hyperbola to demonſtrate thoſe of the circle—Theſe demonſtrations are from the properties of the circle and the theorems for combinations.

The theorems hitherto mentioned are more particularly applicable to the conſtruction of trig. tables and the reſolution of certain equations—In conſequence of the great advances that have been made in phyſical aſtronomy ſince the time of Sir Iſaac Newton, it has been found neceſſary for facilitating the calculation of particular fluents to expreſs the powers of the ſine and coſine in terms of the ſines and coſines of multiple arcs, and theorems for this purpoſe have been given by ſeveral authors. They have all however either deduced the general law from obſervation without demonſtration, or generally demonſtrated it by help of impoſſible logarithms—The demonſtrations

demonſtrations here given are general, and deduced from the circle by help of the doctrine of combinations.

As the hyperbola has been ſo frequently uſed to demonſtrate properties of the circle, I have ſubjoined a theorem by which the connection of multiple circular areas, and multiple hyperbolic areas is more fully apparent than by any other that I have met with, and from whence by the doctrine of combinations, theorems may be deduced for hyperbolic areas ſimilar to thoſe of the circle.

I. Theorem. Let s and c be the ſine and coſine of any arc a, then, radius being unity, and n any whole number,

1. The ſine of $na = nc^{n-1}s - \frac{n.\overline{n-1}.\overline{n-2}}{1.\ 2.\ 3}c^{n-3}s^{3} + \&c.$

2. The coſine of $na = c^{n} - \frac{n.\overline{n-1}}{1.\ 2}c^{n-2}s^{2} + \&c.$

In each the powers of s increaſe by 2, and thoſe of c diminiſh by 2, till the laſt becomes 1 or 0. In the ſine the coefficient of $c^{n-v}s^{v} = \pm\frac{n.\overline{n-1}.\overline{n-2}\ \text{- - (to } v \text{ terms}}{1.\ 2.\ 3\ \text{- - -}\ v}$ + when $\frac{v-1}{2}$ is even and—when odd. And in the coſine the coefficient of $c^{n-v}s^{v} = \pm\frac{n.\overline{n-1}.\ \text{(to } v \text{ terms}}{1.\ 2.\ 3.\ v}$ + when $\frac{v}{2}$ is even and — when odd.

Demonſtration

Demonftration—Let a, a', a'', a''', &c. reprefent any arcs
s, s', s'', s''', their fines
c, c', c'', c''', their cofines

Then by the common theorem for the fine and cofine of the fum of two arcs,

$$\left.\begin{array}{l}\text{The fine}\\ \text{The cofine}\end{array}\right\}\text{ of } a + a' = \left\{\begin{array}{l} s c' + s' c \\ c c' - s s' \end{array}\right.$$

$$\left.\begin{array}{l}\text{The fine}\\ \text{The cofine}\end{array}\right\}\text{ of } a + a' + a'' = \left\{\begin{array}{l} s c' c'' + s' c c'' + s'' c' c - s s' s'' \\ c c' c'' - c s' s'' - c'' s s' \end{array}\right.$$

&c. &c.

The following obfervations may be readily made by confidering the way which in thefe fucceffive values are formed.

1. In both fine and cofine of the fum of n arcs ($a + a' + a''$ &c. the number of factors $s s' - - - c c'$ in any term is equal to n and that the fines s, s', s'', &c. and alfo the cofines c, c', c'', &c. are concerned exactly alike in the whole quantity.

2. In the fine of the fum of n arcs ($a + a' +$ &c.) the greateft number of cofines c, c', c'', &c. together in any term $= n - 1$. This number diminifhes by 2, and confequently the number of s, s', &c. increafes by 2.

3. In

3. In the cofine of the fum of n arcs the greateft number of c, c', c'', &c. in any term $= n$. the next lefs number $n-2$, &c. and confequently the number of s, s', &c. increafes by 2.

4. With refpect to the figns of the different products—In the fine of n arcs $(a + a' + a'' + \&c.)$ when 1, 5 or $4p + 1$ (p being any number) s, s', s'' &c. are united together, the fign is + otherwife —. In the cofine of n arcs when 2, 6, 10 or $2p$ (p being odd) s, s', s'' are united together the fign will be — otherwife +.

5. In no term can the fine and cofine of the fame arc occur.

6. In any term $s\,s'\,s''$ - - - $c\,c'\,c''$ - - - whether of the fine or cofine if m be the number of the cofines and confequently $m-n$ the number of the fines: then, becaufe each of the quantities s, s' &c. and alfo c, c' &c. are concerned exactly alike in the fine of the fum of n arcs $(a + a' + a'' + \&c.)$, and alfo in the cofine of the fum of n arcs $(a + a' + a'' + \&c.)$ and likewife becaufe the fine and cofine of the fame arc cannot occur in the fame term, it follows that the number of terms $s\,s'\,s''$ (m terms) - - - $c\,c'\,c''$ - - - ($\overline{m-n}$ terms) = the number of combinations of n things taken m together $= \dfrac{n.\ \overline{n-1}.\ \overline{n-2}\ \text{- - -}\ \overline{n-m-1}}{1.\ 2.\ 3.\ \text{- - -}\ m}$.

From

From theſe obſervations it immediately follows, if a, a', a'', &c. are all equal, that the ſine of $n a = n c^{n-1} s - \frac{n.\overline{n-1}.\overline{n-2}}{1.\ 2.\ 3.} c^{n-3} s^{3} +$ &c. and that the coſine of $n a = c^{n} - n \frac{\overline{n-1}}{1.\ 2} c^{n-2} s^{2} +$ &c. and alſo that the general terms are as ſtated in the theorem. Q. E. D.

II. Theorem. 1. The coſine of $n a = 2^{n-1} c^{n} - n.\ 2^{n-3} c^{n-2} + \frac{n.\ \overline{n-3}}{1.\ 2} 2^{n-5} c^{n-4}$ &c. to be continued by ſucceſſively diminiſhing the index of c by 2 till it becomes 1 or 0, and affixing to c^{n-u} the coeff.

$$\pm 2^{n-u-1} \cdot \frac{n.\ \overline{n-u+1}.\ \overline{n-u+2}.\ -\ -\ \text{to}\ \frac{u}{2}\ \text{terms}}{1.\quad 2.\quad 3\ -\ -\ -\ \frac{u}{2}}$$

of which the ſign is $+$ when $\frac{u}{2}$ is even, and $-$ when odd.

2. The ſine of $n a = \overline{2^{n-1} c^{n-1} - \overline{n-2}.\ 2^{n-3} c^{n-3} + \&c.} : \sqrt{1-c^{2}}$ continued by diminiſhing the index of c by 2 till it becomes 1 or 0, and affixing to c^{n-u} the coefficient

$$+\ 2^{n-u} \times \frac{\overline{n-u+1}.\ \overline{n-u+2}.\ -\ -\ (\frac{u-1}{2}\ \text{terms})}{1.\quad 2.\quad -\quad -\quad \frac{u-1}{2}}$$

of which the ſign is $+$ when $\frac{u+1}{2}$ is odd and $-$ when even.

Demonstr.

Demonstr. By ſubſtituting in the values of the ſine and coſine of $n\,a$ found by the laſt theorem, for n ſucceſſively 2, 3, 4, &c. and exterminating s it may be conjectured that the general terms of the ſine and coſine will be as here ſtated. That this conjecture is true appears in the following manner:

Let $B\,c^{n-1-u}$ be a term in the coſine of $\overline{n-1}\,a$, and $C\,c^{n-u}\,\sqrt{1-c^2}$, and $D\,c^{n-u-2}\,\sqrt{1-c^2}$ terms in the ſine of $\overline{n-1}\,a$: and that the latter terms will be of this form appears from the former theorem. Applying the common theorem for the ſine and coſine of the ſum of two arcs, it readily appears that the coeff. of c^{n-u} in the coſine of $n\,a = B - C + D$.

Now ſuppoſing the theorem generally true and ſubſtituting in the general terms for n, $\overline{n-1}$ and for u ſubſt. u, $u-1$ and $u+1$ ſucceſſively, the reſult is

$$B = \pm 2^{n-u-2} \times \frac{\overline{n-1}\cdot\overline{n-u}.\,\overline{n-u+1}\ \text{- - - - to}\ \frac{u}{2}\ \text{terms}}{1.\quad 2.\quad 3\ \text{- - - - to}\ \frac{u}{2}\ \text{terms}}$$

$$-C = \pm 2^{n-u} \times \frac{\overline{n-u+1}.\,\overline{n-u+2}\ \text{- - - - to}\ \frac{u}{2}-1\ \text{terms}}{1.\ 2.\ 3.\ \text{- - - - - - to}\ \frac{u}{2}-1\ \text{terms}}$$

D=

$$D = \pm 2^{n-u-2} \times \frac{\overline{n-u-1}\,.\,\overline{n-u}\;\text{- - - - to}\;\frac{u}{2}\;\text{terms}}{1.\quad 2.\quad 3.\;\text{- - to}\;\frac{u}{2}\;\text{terms}}$$

$$\therefore\; B-C+D = \left\{\begin{array}{l} +\,2^{n-u-1} \times \dfrac{\overline{n-1}\,.\,\overline{n-u}}{u} \\ \pm\,2^{n-u-1} \times 2\,.\,\overline{n-\frac{u}{2}-1} \\ \pm\,2^{n-u-1} \times \dfrac{\overline{n-u-1}\,.\,\overline{n-u}}{u} \end{array}\right\} \times \frac{\overline{n-u+1}\,.}{1.\quad 2.\;\text{-}}$$

$$\frac{\overline{n-u+2}\;\text{- to}\;\overline{\frac{u}{2}-2}\;\text{terms}}{\text{- - - -}\;\overline{\frac{u}{2}-1}} =$$

$$\pm\,2^{n-u-1} \times \frac{n\,.\,\overline{n-u+1}\,.\,\overline{n-u+2}\;\text{- - to}\;\frac{u}{2}\;\text{terms.}}{1.\;2.\quad 3.\;\text{- - -}\;\frac{u}{2}}$$

LET alfo $G\,c^{n-u-1}\,s$ be a term in the fine of $\overline{n-1}.\,a$, and let $H\,c^{n-u}$ be a term in the cofine of $\overline{n-1}\;a$, and it readily appears that $G+H=$ coeff. of the term $c^{n-u}\,s$ in the fine of $n\,a$. Now fuppofing the general term of the fine truly expreffed.

 G =

$$G = \pm\, 2^{n-u-1} \times \frac{\overline{n-u}\cdot\overline{n-u+1}\ \text{- - - to}\ \frac{u-1}{2}\ \text{terms}}{1.\ 2.\ 3.\ \text{- - - - - - to}\ \frac{u-1}{2}\ \text{terms}}$$

$$H = \pm\, 2^{n-u-1} \times \frac{\overline{n-1}\,.\ \overline{n-u+1}\ \text{- - - - to}\ \frac{u-1}{2}\ \text{terms}}{1.\ 2.\ 3\ \text{- - - - - -}\ \frac{u-1}{2}\ \text{terms}}$$

$$G + H = +\, 2^{n-u} \times \frac{\frac{1}{2} \times \overline{2\,n-u-1}\,.\ \overline{n-u+1}\ \text{- to}\ \frac{u-1}{2}\ \text{terms}}{1.\ 2.\ 3\ \text{- -}\ \frac{u-1}{2}\ \text{terms}}$$

$$= \pm\, 2^{n-u} \times \frac{\overline{n-u+1}\,.\ \overline{n-u+2}\ \text{- - to}\ \frac{u-1}{2}\ \text{terms}}{1.\ 2.\ 3\ \text{- - to}\ \frac{u-1}{2}\ \text{terms.}}$$

Hence it appears that if the general terms are rightly expreſſed for the ſine and coſine of $\overline{n-1}\ a$, they are alſo rightly expreſſed for the ſine and coſine of $n\,a$, conſequently if they are true in the inferior values of n they are true in the ſuperior, but they are true in the inferior ∵ &c. &c.

III. Cor. If the ſeries be arranged in a contrary order:

1. When

1. When n is *even* the cofine of $na = \pm 1 \mp \frac{n^2 c^2}{1.\,2} \pm \frac{n^2.\,\overline{n^2 - 2^2}}{1.2.3.4.} c^4$ $\mp$ &c. and the general term is $\pm \frac{n^2.\,\overline{n^2 - 2^2}.\,\overline{n^2 - 4^2} \text{ - to } \frac{v}{2} \text{ terms}}{1.\ 2.\ 3 \text{ - - } v} c^v$ where v is always even. When n is of the form $2p$, (p being any *odd* number the fign will be + or — according as $\frac{v}{2}$ is odd or even and when n is of the form $4p$, (p being any number) it will be + or — according as $\frac{v}{2}$ is *even* or *odd.*

2. When n is *odd*, the cofine of $n\,a = \pm n\,c \mp \frac{n.\,\overline{n^2 - 1}}{1.\ 2.\ 3} c^3 \pm$ &c. and the general term is $\pm \frac{n.\,\overline{n^2 - 1}.\,\overline{n^2 - 3^2} \text{ - to } \frac{v + 1}{2} \text{ terms}}{1.\ 2.\ 3 \text{ - - } v} c^v$ where v is always odd. When n is of the form $4p + 1$ the fign will be + or — according as $\frac{v + 1}{2}$ is odd or even, when of the form $4p + 3$ it will be + or — according as $\frac{v + 1}{2}$ is even or odd. Each feries is to be continued till the coefficient becomes $= 0$.

Dem. The general term of the cofine of $n\,a$.

$= \pm$

$$= \pm 2^{n-u-1} \times \frac{u\,.\,\overline{n-u+1}\,.\,\overline{n-u+2} \; - \; - \; \frac{u}{2}\text{ terms}}{1\,.\;2\,.\;3 \; - \; - \; \frac{u}{2}\text{ terms}}\, c^{n-u}$$

or fubftituting for u, $n-v$, the coeff. becomes

$$\frac{\pm\, n\,.\,\overline{v+1}\,.\,\overline{v+2} \; - \; \frac{n+\overline{v-4}}{2}\,.\,\frac{n+\overline{v-2}}{2}}{1\,.\;2\;3 \; - \; - \; \frac{n-v}{2}}\, 2^{v-1}$$

$$= \pm \frac{n\,\frac{n-\overline{v-2}}{2}\,\frac{n-\overline{v-4}}{2} \; - \; \frac{n+\overline{v-4}}{2}\,.\,\frac{n+\overline{v-2}}{2}}{1\,.\;2\,.\;3 \; - \; v}\, 2^{v-1}$$

$$= \pm \frac{n\,.\,\overline{n-\overline{v-2}}\,.\,\overline{n-\overline{v-4}} \; - \; - \; \overline{n+\overline{v-4}}\,.\,\overline{n+\overline{v-2}}}{1\,.\;2\,.\;3 \; - \; - \; v}$$

1. When n is even and $\because$ v even it is of this form

$$+\frac{n\,.\,\overline{n-\overline{v-2}} \; - \; - \; \overline{n-2}\,.\,n\,.\,\overline{n+2} \; - \; - \; n+\overline{v-2}}{1\,.\;2\,.\;3 \; - \; - \; v}$$

$$= + \frac{n^2\,.\,\overline{n^2-2^2}\,.\,\overline{n^2-4^2} \; - \; - \; \text{to } \frac{v}{2}\text{ terms}}{1\,.\;2\,.\;3 \; - \; - \; v}$$

The fign is + or — according as $\frac{u}{2}$ or $\frac{n-v}{2}$ is even or odd.

∴ If

∴ If n be of the form $2p$ (p being odd) the ſign is + or — according as $\frac{2p-v}{2}$ is even or odd ∴ as $\frac{v}{2}$ is odd or even. If n be of the form $4p$ then it is + or — as $\frac{4p-v}{2}$ is even or odd and ∵ as $\frac{v}{2}$ is even or odd.

2. When n is odd and ∵ v odd the gen. coeff. becomes of this form $\pm\frac{n.\overline{n-v-2}\ -\ -\ \overline{n-1}.\overline{n+1}\ -\ -\ n+\overline{v-2}}{1.\ 2.\ 3\ -\ -\ -\ v}$

$$=\pm\frac{\overline{2\quad 2}\ \ \overline{2\quad 2}\qquad\qquad\qquad\qquad}{\begin{array}{c}n.\,n-1\,.\,n-3\ -\ -\ \text{to } \frac{v+1}{2} \text{ terms.}\\ \hline 1.\ 2.\ 3\ -\ -\ -\ v\end{array}}$$

The ſign is + or — according as $\frac{n}{2}$ or $\frac{n-v}{2}$ is even or odd.

∴ If n be of the form $4p+1$ it is + or — as $\frac{4p+1-v}{2}$ or $\frac{4p+2-\overline{v+1}}{2}$ is even or odd or ∴ as $\frac{v+1}{2}$ is odd or even. If n be of the form $4p+3$, it is + or — as $\frac{4p+3-v}{2}$ or $\frac{4p+4-\overline{v+1}}{2}$ or ∴ as $\frac{v+1}{2}$ is even or odd. Whence &c. &c.

IV. Theorem.

IV. THEOREM. 1. When n is any even number The fine of $n a = \pm 2^{n-1} s^{n-1} \mp 2 . \overline{n-2}^{\,n-3} s^{n-3} \pm$ &c. : $\sqrt{1-s^2}$ to be continued by diminishing the index of s by 2 till it becomes unity. The upper figns take place when n is of the form $2p$ (p being odd) and the lower when it is of the form $4p$ (p being any number).

THE general term is $\pm \dfrac{\overline{n-u+1} . \overline{n-u+2} \text{ - to } \frac{u-1}{2} \text{ terms}}{1 . 2 . 3 \text{ - } \frac{u-1}{2}}$

$\times 2^{n-u} s^{n-u} \times \sqrt{1-s^2}$: $\left.\begin{matrix} + \text{ when } \frac{u+1}{2} \text{ is odd} \\ - \text{ when } \frac{u+1}{2} \text{ is even} \end{matrix}\right\}$ and n of the form $2p$ (p being odd.)

$\left.\begin{matrix} + \text{ when } \frac{u+1}{2} \text{ is even} \\ - \text{ when } \frac{u+1}{2} \text{ is odd} \end{matrix}\right\}$ and n of the form $4p$ (p being any number)

2. WHEN n is any odd number, the fine of $n a = \pm 2^{n-1} s^{n} \mp n 2^{n-3} s^{n-2} +$ &c. to be continued by diminifhing the index of s by 2 till it becomes unity. The upper figns take place when n is of the form $4p + 1$, and the under when of the form $4p + 3$.

THE

The general term is $\pm \dfrac{n.\overline{n-u+1}.\overline{n-u+2}\ \text{- to}\ \frac{u}{2}\ \text{terms}}{1.\ 2.\ 3\ \text{- -}\ \frac{u}{2}} \times$

$2^{n-u-1} s^{n-u}.$

$\left.\begin{array}{l} + \text{ when } \frac{u}{2} \text{ is even} \\ - \text{when } \frac{u}{2} \text{ is odd} \end{array}\right\}$ and n of the form $4p+1$.

$\left.\begin{array}{l} + \text{ when } \frac{u}{2} \text{ is odd} \\ - \text{ when } \frac{u}{2} \text{ is even} \end{array}\right\}$ and n of the form $4p+3$.

Demon. The general term of the fine of $n \times \overline{Q-a} =$ (II)

$$= \pm \frac{\overline{n-u+1}.\ \overline{n-u+2}\ \text{- -}\ \text{to}\ \frac{u-1}{2}\ \text{terms}}{1.\ 2.\ 3\ \text{- -}\ \frac{u-1}{2}}\ 2^{n-u} . \overline{cs, Q-a}|^{n-u}$$

$\times\ s, \overline{Q-a},$ where Q is a quad.

1. Let n be of the form $2p$, p being odd. The fine of $2p \times \overline{Q-a} = s, \overline{2p-2\,Q + 2\,Q - 2pa} =$ (becaufe $\overline{2p-2}$. Q is a multiple of the circumference) fine $2\,\overline{Q-2pa} = s, 2pa$ ∴ when n is of the form $2p$, p being odd the general term of the fine of $na =$

 +

$$\pm \frac{\overline{n-u+1}\,.\,\overline{n-u+2} \;\cdots\; \text{to } \frac{u-1}{2} \text{ terms}}{1\,.\,2\,.\,3 \;\cdots\; \frac{u-1}{2}} \; 2^{n-u} \; \overline{s,a}^{\,n-u} \times cs\; a,$$

$+$ when $\frac{u+1}{2}$ is odd and — when even.

Let n be of the form $4p$, p being any number.

The fine of $4p \times \overline{Q-a}$ = (becaufe $4p\,Q$ is a multiple of the circumference) = fine of $-4pa = -s,\ 4pa$ $\therefore$ when n is of the form $4p$ the general term of the fine of $na = \pm$

$$\pm \frac{\overline{n-u+1} \;\cdots\; \text{to } \frac{u-1}{2} \text{ terms}}{1\,.\,2 \;\cdots\; \frac{u-1}{2}} \times 2^{n-u} \; \overline{s,a}^{\,n-u} \times cs\; a$$

— when $\frac{u+1}{2}$ is odd and $+$ when even.

2. When n is odd.

The general term of the cofine of $n \times \overline{Q-a} =$ (II)

$$= \pm \frac{n\,.\,\overline{n-u+1}\,.\,\overline{n-u+2} \;\cdots\; \text{to } \frac{u}{2} \text{ terms}}{1\,.\,2\,.\,3 \;\cdots\; \frac{u}{2}} \; 2^{n-u-1} \; cs\; \overline{Q-a}^{\,n-u}$$

Let n be of the form $4p+1$.

The cofine of $\overline{4p+1} \times \overline{Q-a} = cs$ [illegible] $\overline{4p+1}\,a = cs,\ Q - \overline{4p+1}\,a$ = fine $\overline{4p+1}\,a$ $\therefore$ when n is of the form $\overline{4p+1}$. the

the gen. term of the ſine of $n\,a =$

$$\pm\frac{n.\,\overline{n-u+1}.\,\overline{n-u+2} \quad - \quad - \quad \text{to } \frac{u}{2} \text{ terms}}{1.\,2.\,3 \quad - \quad - \quad - \quad - \quad \frac{u}{2}}$$

$2^{n-u-1}\,\overline{s,\,a}^{\,n-u}$ + when $\frac{u}{2}$ is even and — when odd.

Let n be of the form $4p + 3$.

The coſine of $\overline{4p+3} \times \overline{Q-a} = cs$ of $\overline{3\,Q - \overline{4p+3}\,a} =$ (becauſe adding or ſubtracting $\frac{1}{2}$ the circumference changes the ſign of the coſine) $= - cs$ of $\overline{Q - \overline{4p+3}}\,a = - s.$ of $\overline{4p+3}\,a$.

∴ When n is of the form $\overline{4p+3}$ the general term of the ſine of

$$n\,a = \pm\frac{n.\,\overline{n-u+1}.\,\overline{n-u+2} \quad - \quad \text{to } \frac{u}{2} \text{ terms}}{1.\,2.\,3. \quad - \quad - \quad \frac{u}{2}}\; 2^{n-u-1}\,\overline{s\,a}^{\,n-u}$$

— when $\frac{u}{2}$ is even and + when odd. Whence the truth of the theorem will eaſily appear.

V. Cor. If the ſeries be arranged in a contrary order.

The ſine of $n\,\mathrm{A} = n\,s - \frac{n.\,\overline{n^2-1^2}}{1.\,2.\,3}\,s^3 +$ &c. when n is any odd

number; and the sine of $n\,a = n\,s - \frac{n.\,\overline{n^2-2^2}}{1.\,2.\,3}s^3 + \&c.$ $\times \sqrt{1-s^2}$ when n is any even number.

In the former case the general term is

$$\pm \frac{n.\,\overline{n^2-1}.\,\overline{n^2-3^2}\quad - \quad - \quad \text{to } \frac{v+1}{2} \text{ terms}}{1.\,2.\,3 \quad - \quad - \quad - \quad \text{to } v \text{ terms}} \times s^v$$

v being always odd, + when $\frac{v+1}{2}$ is odd and — when even. In the latter case the general term is $\pm$

$$\frac{n.\,\overline{n^2-2^2}.\,\overline{n^2-3^2}.\quad \text{to } \frac{v+1}{2} \text{ terms}}{1.\,2.\,3 \quad - \quad - \quad \text{to } v \text{ terms}}\, s^v \times \sqrt{1-s^2},$$ + when $\frac{v}{2}$ is odd and — when even.

This Cor. may be deduced from the theorem in the same manner as the Cor. Art. III. was deduced from its theorem.

Theorems for the Powers of the Sines and Cosines.

VI. Theorem. If c be the cosine of the arc a and rad. unity then n being any whole positive number.

$\overset{n}{c} = \overline{\tfrac{1}{2}}^{\,n-1} \times : cs\,n\,a + n.\,cs\,\overline{n-2}\,a + \&c.$ cont. to $\frac{n+1}{2}$ terms when n is odd and when n is even to $\frac{1}{2}n+1$ taking only $\frac{1}{2}$ the

last

laſt term. The general term is $\frac{n.\overline{n-1}.\overline{n-2}\ \text{- -}\ \text{to } m \text{ terms}}{1.\ 2.\ 3\ \text{- -}\ m}$ $cs\,\overline{n-2\,m\,a}$.

Dem. Let a, a', a'', &c. repreſent any arcs
c, c', c'', &c. their coſines

Then by trig. $cs, a \times 2\, cs, a' = cs, \overline{a + a'} + cs, \overline{a - a'}$ and in like manner $cs, a \times 2\, cs, a' \times 2\, cs, a'' = \overline{cs, \overline{a + a'} + cs, \overline{a - a'}} \times 2\, cs, a'' = cs, \overline{a + a' + a''} + cs, \overline{a + a' - a''} + cs, \overline{a - a' + a''} + cs, \overline{a - a' - a''}$, &c. &c. and it is evident that to multiply by twice the coſine of any arc it is only neceſſary to encreaſe and diminiſh each of the former quantities $a + a' +$ &c. $a - a'$, &c. by that arc, and take the ſum of the coſines of the arcs ſo encreaſed and diminiſhed : therefore becauſe in the product of the coſines of a, $2\,a'$, $2\,a''$, &c. all the arcs a', a'', &c. muſt be involved exactly alike, it follows that $2^{n-1} \times cs, a \times cs, a' \times cs, a'' \times$ &c. $=$ ſum of the coſines of all the arcs formed by adding to a each combination of the $\overline{n-1}$ arches a', a'', &c. taken poſitively or negatively. Hence by the theorems for combinations, there will be

1. term the coſine of $a + a' + a'' +$ &c.

$\overline{n-1}$

$\overline{n-1}$ terms the cofine (fum $\overline{n-1}$ arcs — 1 arc) (B)

$\frac{\overline{n-1}\ \overline{n-2}}{1.\ 2.}$ terms the cofine (fum $\overline{n-2}$ arcs — fum 2 arcs) (C)

$\frac{\overline{n-1}.\ \overline{n-2}\ -\ \overline{n-m}}{1\ \ 2.\ -\ -\ m}$ terms the cofine (fum of $\overline{n-m}$ arcs — fum m arcs) (H)

$\frac{\overline{n-1}\ -\ -\ \overline{n-m-1}}{1.\ 2\ -\ \overline{m-1}}$ terms the cofine (fum m arcs — fum $\overline{n-m}$ arcs) (H′)

$\overline{n-1}$ terms the cofine (fum 2 arcs — fum $\overline{n-2}$ arcs) (C′)

1. term the cofine (1 arc (a) — Sum $\overline{n-1}$ arcs) B′.

Now if the arcs be all taken equal, all the Bs are equal to each other, all the Cs, &c. &c. and alfo $B = -B'$, $C = -C'$ &c. &c. and confequently $cs, B = cs, B'$, $cs, C = cs, C'$, &c. &c.

$$\because\ \frac{\overline{n-1}.\ -\ \overline{n-m}}{1.\ 2.\ -\ m}\, cs\, H + \frac{\overline{n-1}\ \ -\ \ \overline{n-m-1}}{1.\ 2.\ 3\ -\ \overline{m-1}}\, cs\, H'$$

$$= \frac{n.\ \overline{n-1}\ -\ -\ \overline{n-m-1}}{1.\ 2\ -\ -\ m}\, cs, \overline{n-2\ m\ a}.$$

WHENCE

WHENCE $\overset{n}{c} = \overline{\tfrac{1}{2}}\rceil^{n-1} \times cs\, n\, a + n.\, cs\, \overline{n-2}\, a + \frac{n.\, \overline{n-1}}{1.\ 2}\, cs\, \overline{n-4}\, a + \&c.$ continued to $\frac{n+1}{2}$ terms when n is odd: but when n is even there will be a middle term $\frac{\overline{n-1}.\ \overline{n-2}\ -\ -\ \frac{n}{2}}{1.\ 2\ -\ -\ \frac{1}{2}n} \times cs,\ \overline{n-2.\,\frac{n}{2}}\, a$

$$= \frac{n.\ \overline{n-1}\ -\ -\ \text{to}\ \frac{n}{2}\ \text{terms}}{2.\ 1.\ 2.\ 3\ -\ -\ \frac{1}{2}n} \times cs\ o\, a \because \text{in this case}$$

$$\overset{n}{c} = \overline{\tfrac{1}{2}}^{\,n-1} : \times cs\ n\, a + n.\ cs\ \overline{n-2}\, a + \&c.\ \text{to}\ \frac{n}{2}\ \text{terms} + \tfrac{1}{2} \times \frac{n.\ n-1\ -\ -\ \frac{n}{2}\ \text{terms.}}{1.\ 2.\ -\ -\ \frac{n}{2}}$$

VII. THEOREM. 1. When n is any odd number, and s the sine of any arc. a, rad. being unity, $\overset{n}{s} = \overline{\tfrac{1}{2}}\rceil^{n-1} \times \pm s,\ n\, a \mp n.\ s,\ \overline{n-2}\, a \pm \frac{n.\ n-1}{1\ \ 2.}\ s,\ \overline{n-4}\, a \mp \&c.$ continued to $\frac{n-1}{2}$ terms &c. the upper signs taking place when n is any odd number of the form $4p + 1$, and the lower when of the form $4p + 3$.

THE

THE general m^{th}. term is $\pm \frac{n.\,\overline{n-1}\ \text{-}\ (\text{to } m \text{ terms}}{1.\ 2.\ \text{-}\ m}\, s,\ \overline{n-2m}\, a$

$+$ when m is even, $-$ when m is odd } and n of the form $4p+1$.

$+$ when m is odd, $-$ when m is even } and n of the form $4p+3$.

2. WHEN n is any even number.

$\overset{n}{s} = \overline{\tfrac{1}{2}}\rceil^{n-1} \times \pm cs\ na \mp n.\ cs\ \overline{n-2}\, a \pm$ &c. (to $\frac{n}{2}$ terms) $\pm$

$\overline{\tfrac{1}{2}}\rceil^{n-1} \times \frac{n.\ \overline{n-1}.\ \overline{n-2}\ \text{-}\ \frac{1}{2}n \text{ terms}}{2.\ 1.\ 2.\ 3\ \text{-}\ \text{-}\ \frac{1}{2}n}$. The upper signs take place when n is of the form $4p$, and the lower when of the form $2p$, p being any odd number. The m^{th}. term is

$\pm \frac{n.\ \overline{n-1}\ \text{-}\ (m \text{ terms}}{1.\ 2.\ 3\ \text{-}\ (m \text{ terms})}\, cs\ \overline{n-2m}\, a$

$+$ when m is odd, $-$ when m is even } and n of the form $2p$, p being odd.

$+$ when m is even, $-$ when m is odd } and n of the form $4p$.

DEM. Let Q = a quadr. then (VI) $\overset{n}{cs.\ \overline{Q-a}\rceil} = \overline{\tfrac{1}{2}}\rceil^{n-1} \times cs\ n.\ \overline{Q-a}$ $+$ &c. and the general m^{th}. term is $\frac{n.\ \overline{n-1}.\ \text{-}\ \text{to } m \text{ terms}}{1.\ 2.\ 3\ \text{-}\ m \text{ terms}}$ $cs\ \overline{n-2m}.\ \overline{Q-a}$.

1. 1st

1. 1ſt. When n is of the form $4p+1$, ſubſt. for n, $4p+1$ $cs\ \overline{n-2\,m}.\ \overline{Q-a} = cs,\ \overline{\overline{4\,p-2\,m}\,Q + Q - \overline{n-2\,m}\,a} =$ (becauſe adding or ſubtracting the circumference makes no alteration in the value of the coſine and adding or ſubtracting $\frac{1}{2}$ the circumference changes the ſign of the coſine) $\pm\ cs\ \overline{Q - \overline{n-2\,m}\,a} = \pm\ s,\ \overline{n-2\,m}\,a$ + when m is even and — when odd.

1. 2. When n is of the form $4p+3$, ſubſt. for n, $4p+3$, $cs,\ \overline{n-2\,m}.\ \overline{Q-a} = cs,\ \overline{\overline{4\,p+3-2\,m}\,Q - \overline{n-2\,m}\,a} = \pm\ s,\ \overline{n-2\,m}\,a$, + when m is odd and — when even.

2. 1. When n is even of the form $2p$, p being odd, ſubſt. for n $2p$, $cs\ \overline{n-2\,m}.\ \overline{Q-a} = cs\ \overline{\overline{2\,p-2\,m}\,Q - \overline{n-2\,m}\,a} = \pm\ cs\ \overline{n-2\,m}\,a$, + when m is odd and — when even.

2. 2. When n is of the form, $4p$; ſubſtituting for n, $4p$, $cs\ \overline{n-2\,m}.\ \overline{Q-a} = cs\ \overline{\overline{4\,p-2\,m},\ Q - \overline{n-2\,m},\ a} = \pm\ cs\ \overline{n-2\,m},\ a$ + when m is even and — when odd.

Whence ſubſtituting in the general term for the $cs,\ \overline{Q-a}$, the $s,\ a$ and for $cs,\ \overline{n-2\,m}\,a$, the values above found, the truth of the theorem is evident.

Properties of the Equilateral Hyperbola.

VIII. Theorem. Let a, a', a'' reprefent abfciffas meafured from the centre on the axis of an equilateral hyperbola, and o, o', o'' correfponding ordinates: let alfo the hyperbolic area contained by the femi axis (= unity), diftance from the centre to the extremity of the arc, and the arc, the abfciffa of which is a'' and ordinate o'', be equal to the fum of the areas contained in the fame manner by the femi axis, dift^{s}. and arcs the abfciffas and ordinates of which are a, a' and o, o': then will $a'' = a\,a' + o\,o'$ and $o'' = a\,o' + a'\,o$.

Dem. Let the area ACV (fee fig.) = ECV + BCV, let the double ordinates FEe, bGB, aHA be produced to meet the affymptote Cw' x' y' NYX m n Wp, and let fall the perp^{s}. aw', bx', ey', VN, EY, BX, AW. Becaufe ACV = EVC + BCV and becaufe (by prop. hyperb.) CVN = ECY = BCX = ACW $\because$ VNEY + VNBX = VNAW or VNEY = BAWX: and it has been proved by many writers on conics that when thefe areas are equal

$$CN : CY :: CX : CW$$
$$\text{or } VN : EY :: BX : AW$$

Whence it follows that

$$CV : Em :: Bn : Ap$$
$$\text{or } 1 : a - o :: a' - o' : a'' - o''$$

in

in like manner it may be ſhewn

that $CV :: em :: bn : ap$

or $1 : a + o :: a' + o' : a'' + o''$

hence $a'' - o'' = aa' - ao' - a'o + oo'$

and $a'' + o'' = aa' + ao' + a'o + oo'$

and $\because a'' = aa' + oo'$ and $o'' = ao' + a'o$. Q. E. D.

FROM the ſimilitude between theſe theorems and thoſe for the ſine and coſine of the ſum of two circular arcs, it is unneceſſary to point out how every thing may be deduced for multiple hyperbolic areas in the ſame manner as was done for multiple circular arcs.